

## Lecture Notes Handout 7

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Semester I 1999/2000

### EC211

## Introduction to Mathematical Economics

### Topic 2 Calculus I (cont'd)

- Univariate calculus
  - The derivative and differentiation
  - Basic rules of differentiation
  - 2nd derivatives, and higher order derivatives
- Multivariate calculus
  - Partial derivatives/differentiation
  - Higher order partial derivatives

### Another example

Given a demand curve, to find marginal revenue:

Our demand curve is:

$$Q = 50 - \frac{1}{2}P$$

We can re-write this as an inverse demand function:

$$P = 100 - 2Q$$

We know that total revenue is simply:

$$\begin{aligned} TR &= PQ \\ &= (100 - 2Q)Q \\ &= 100Q - 2Q^2 \end{aligned}$$

and so that marginal revenue is:

$$\frac{dTR}{dQ} = 100 - 4Q$$

## Rule for products of functions

Instead of being the sum/difference of functions,  $y = f(x)$  might be the product of functions (of  $x$ ). We state the rule for finding  $\frac{dy}{dx}$  where  $y$  is the product of two functions  $v$  and  $u$ :

Let

$$v = g(x) \quad \text{and} \quad u = h(x)$$

and

$$y = vu \equiv g(x)h(x)$$

Our rule is:

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$$

or, using the alternative notation:

$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

## Example of the product rule

Given

$$v = g(x) = 4x^3 + 2x^4 \quad \text{and} \quad u = h(x) = 3x^2$$

$$\text{and} \quad y = f(x) = vu = (4x^3 + 2x^4)(3x^2)$$

Then:

$$\begin{aligned} \frac{dy}{dx} &= (4x^3 + 2x^4) \frac{d(3x^2)}{dx} + (3x^2) \frac{d(4x^3 + 2x^4)}{dx} \\ &= (4x^3 + 2x^4)6x + (3x^2)(12x^2 + 8x^3) \\ &= 24x^4 + 12x^5 + 36x^4 + 24x^5 \\ &= 60x^4 + 36x^5 \end{aligned}$$

In this case, we could confirm the result by working out the product  $vu$  directly and taking the derivative:

$$y = vu = (4x^3 + 2x^4)(3x^2) = 12x^5 + 6x^6$$

So that:

$$\frac{dy}{dx} = 60x^4 + 36x^5$$

(This may not always be possible: we might only be given  $v$  and  $u$  in their general form.)

## Rule for Quotients (ratios) of functions

Our function  $y = f(x)$  may take the form of a quotient, i.e the ratio of two functions:

$$y = f(x) = \frac{u}{v} = \frac{h(x)}{g(x)}$$

our quotient rule is:

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

### An example

$$\begin{aligned} \frac{d}{dx} \left( \frac{3x^2 + 4}{x^2 + 2} \right) &= \frac{(x^2 + 2)(6x) - (3x^2 + 4)(2x)}{(x^2 + 2)^2} \\ &= \frac{4x}{(x^2 + 2)^2} \end{aligned}$$

## Rule for functions of functions (the chain rule)

We may have the following case to deal with:

$$y = f(z) \quad \text{where} \quad z = g(x)$$

i.e.  $y$  is a function of a function. To find  $\frac{dy}{dx}$  we use **the chain rule**:

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx}$$

(The  $dz$ 's cancel.)

### An example

$$y = 2z^2 \quad \text{where} \quad z = 3x + 10$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dz} \frac{dz}{dx} = \frac{d(2z^2)}{dz} \frac{d(3x+10)}{dx} \\ &= (4z)(3) = (4)(3x+10)(3) \\ &= 36x + 120 \end{aligned}$$

## Higher Order Derivatives

Up to now, we've had simply a **primitive** or original function  $y = f(x)$  and differentiated this, to find the **first derivative**  $\frac{dy}{dx}$ , which is itself in general a function of  $x$ .

So we might take the derivative of this derivative (applying exactly the same rules we've seen), to get the **second derivative** of the function, denoted as follows:

$$\frac{d^2 y}{dx^2} \quad \text{or} \quad f''(x)$$

So we've seen **first order** and **second order** derivatives. Depending on the context, and the function, we might need third or higher order derivatives. In general, an  $n$ th-order derivative of  $y = f(x)$  is denoted:

$$\frac{d^n y}{dx^n}$$

The first derivative is the slope of the function, the second derivative tells is the slope of the first derivative, and so on. . .

### **An example**

Consider a utility function, where  $U$  is total utility, which depends only  $x$  the quantity of a single good consumed:

$$U = U(x)$$

Typical neo-classical assumptions are that:

$$\frac{dU}{dx} > 0 \quad \text{and} \quad \frac{d^2U}{dx^2} < 0$$

Using the alternative notation:

$$U'(x) > 0 \quad \text{and} \quad U''(x) < 0$$

Equivalently:

- we usually assume positive but diminishing marginal utility;
- the utility function slopes upwards, but gets flatter;
- the marginal utility function is downward sloping.

### Example: total and marginal costs

Given a Total Cost function for a firm:

$$TC = TC(Q) = Q^3 - 12Q^2 + 60Q$$

(where  $Q > 0$ )

Marginal Cost is:

$$MC \equiv \frac{dTC}{dQ} = 3Q^2 - 24Q + 60$$

The slope of the  $MC$  cost function is:

$$\frac{dMC}{dQ} \equiv \frac{d^2TC}{dQ^2} = 6Q - 24$$

This tells us that when  $Q < 4$ ,  $MC$  is falling (since when  $Q < 4$ ,  $\frac{dMC}{dQ} < 0$ ) and when  $Q > 4$ ,  $MC$  is rising, (since when  $Q > 4$ ,  $\frac{dMC}{dQ} > 0$ ).

### Multivariate Calculus

In most economic contexts, we are dealing with functions of more than one variable — multivariate functions. For example:

a utility function:  $U = U(x_1, x_2, x_3)$

where  $x_1$  is the quantity of good 1 consumed, and so on.

a production function:  $Y = F(K, L)$

which regards output  $Y$  as depending on the amount of capital inputs  $K$  and the amount of labour inputs  $L$ .

In applying calculus to such functions, we use essentially the same rules as set out so far, with the proviso that we consider afresh what we might mean by a derivative in this new context.

We use calculus in this context to express the *ceteris paribus* (all other things being equal) practice in economics:

We ask 'what happens to one variable, if we change one factor determining it, *keeping everything else constant?*'

E.g. what is the change in total utility, if we change the quantity of good 1 consumed, but hold the amounts of goods 2 and 3 constant? i.e. what is the marginal utility of good 1?

Or, by how much does total output  $Y$  change, if we increase the amount of labour  $L$ , but leave capital  $K$  constant? i.e. what is the marginal product of labour?

These are derivative ideas, but reflecting this additional idea of keeping all other variables constant, other than the one with respect to which we are differentiating, we refer to them as the **partial** derivatives (of utility with respect to good 1, and of output with respect to labour, in the examples chosen).

The notation looks like this:

$$\frac{\partial U}{\partial x_1} \quad \text{or} \quad U_{x_1} \quad \text{or} \quad MU_{x_1}$$

and

$$\frac{\partial Y}{\partial L} \quad \text{or} \quad F_L \quad \text{or} \quad MPL$$

Note we've used three versions of the Greek letter 'delta': the capital  $\Delta$  (for large-ish changes), small 'delta'  $\delta$ , usually as a parameter, and now another lowercase 'delta'  $\partial$ , read as 'the partial derivative of. . . with respect to. . .'.)

## Partial Differentiation

There is a partial derivative corresponding to each independent variable in our function, i.e. given

$$y = f(x_1, x_2, x_3, \dots, x_n)$$

we could find the  $n$  partial derivatives,

$$\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3}, \dots, \frac{\partial y}{\partial x_n}$$

To do so, in each case, we apply the rules of differentiation as before, but treat all independent variables as constant other than the one with respect to which we are differentiating.

## Example

Given

$$y = 4x_1^3x_2^2$$

Treating  $x_2$  (and therefore  $x_2^2$ ) as constant, and applying the power function rule, as before to get:

$$\frac{\partial y}{\partial x_1} = (3)(4)(x_2^2)(x_1^{3-1}) = 12x_2^2x_1$$

Similarly:

$$\frac{\partial y}{\partial x_2} = (2)(4)(x_1^3)(x_2^{2-1}) = 8x_1^3x_2$$

...in this case, treating terms in  $x_1$  as constant.

## A further example

Given

$$y = 2x_1^2 + 3x_1x_2$$

Note that we are in fact taking the partial derivative of the sum of functions: which as before, is the sum of the (partial) derivatives.

When working out  $\frac{\partial y}{\partial x_1}$ , the first term  $2x_1^2$  presents nothing new.

In the second term  $3x_1x_2$ , we regard *both* 3 and  $x_2$  as constants:

it may be easier to re-write this term as  $(3x_2)x_1$ .

Then apply the power function rule exactly as before to get:

$$\frac{\partial y}{\partial x_1} = 4x_1 + 3x_2$$

Similarly:

$$\frac{\partial y}{\partial x_2} = 0 + 3x_1 = 3x_1$$

(In the case of the first term here, it is equivalent to a constant, and by application of the power function rule (again), this partial derivative is zero.)

## An example of comparative static analysis

In comparative static analysis, we ask 'what happens to an equilibrium when we change one variable, keeping all others constant?'

In a previous example (Notes 3 page 16) we solved a simple macro model to arrive at the following equilibrium value for national income  $Y$ :

$$Y^* = \frac{I_0 + G_0 + a}{(1 - b)}$$

( $b$  was the marginal propensity to consume).

We are now able to frame the question: how does equilibrium national income vary, if we increase government expenditure  $G_0$ ? We are to find:

$$\frac{\partial Y^*}{\partial G_0}$$

This is straightforward, in that we can re-write the expression for  $Y^*$  as:

$$Y^* = \frac{I_0}{(1 - b)} + \frac{G_0}{(1 - b)} + \frac{a}{(1 - b)}$$

and, treating everything other than  $G_0$  as constant, we have

$$\begin{aligned} \frac{\partial Y^*}{\partial G_0} &= 0 + \frac{1}{(1 - b)} + 0 \\ &= \frac{1}{(1 - b)} \end{aligned}$$

This result, is of course, the familiar simple multiplier of a Keynesian closed economy:

$$\frac{1}{(1 - MPC)}$$